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Applied Mathematics Letters 19 (2006) 155–160

Applied Mathematics Letters

www.elsevier.com/locate/aml

A new refined form of Jordan's inequality and its applications

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Received 14 November 2003; received in revised form 9 May 2005; accepted 12 May 2005

Abstract

A new refined form of Jordan's inequality [D.S. Mitrinovic, Analytic Inequalities, Springer-Verlag, 1970; F. Yuefeng, Jordan's inequality, Math. Mag. 69 (1996) 126] is proved and an application of it, together with some numerical results, are given.

Keywords: Jordan's inequality; Increasing functions; Decreasing functions

1. Introduction and main results

Consider Jordan's inequality [1,2]

$$\frac{\sin x}{x} \geq \frac{2}{\pi}, \quad x \in \left(0, \frac{\pi}{2}\right] \tag{1}$$

with equality if and only if $x = \frac{\pi}{2}$. Jordan's inequality and its refinements are important in calculus and trigonometry with applications in the theory of limits [3], approximating Riemann zeta function $\zeta(x)$ [4] and in generalizations of Yang's inequality [5] which, together with its extensions, plays an important role in the theory of distribution of values of functions [5–7]. In a recent paper by Debnath and Zhao [8] it has been shown that

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{12\pi}(\pi^2 - 4x^2), \quad x \in \left(0, \frac{\pi}{2}\right] \tag{2}$$

and

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.05.003

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2), \quad x \in \left(0, \frac{\pi}{2}\right] \tag{3}$$

with equalities if and only if $x = \frac{\pi}{2}$. They also provide some applications based on these strengthened forms of Jordan's inequality. The inequalities (2) and (3) had been considered by a number of other authors at an earlier date [9].

In this paper we will consider a new refined form of Jordan's inequality and an application of it on the same problem considered by Debnath and Zhao [8]. Our main result is given by the following.

Theorem 1.1. *If $x \in \left(0, \frac{\pi}{2}\right]$, then*

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi - 3)}{\pi^3}\left(x - \frac{\pi}{2}\right)^2 \tag{4}$$

with equality if and only if $x = \frac{\pi}{2}$.

Proof. Consider the function $f : [0, \frac{\pi}{2}] \longrightarrow R$ defined by

$$f(x) = 6\sin x - x^2 \sin x - 4x\cos x - 2x.$$

Then we have

$$f'(x) = 2\cos x + 2x\sin x - x^2\cos x - 2$$

and

$$f''(x) = x^2 \sin x.$$

Since $f''(x) > 0$ for all $x \in (0, \frac{\pi}{2})$, this implies that $f'(x)$ is strictly increasing on $(0, \frac{\pi}{2}]$. Hence $f'(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$ since $f'(0) = 0$. Therefore $f(x)$ is strictly increasing. But then $f(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$ since $f(0) = 0$. So

$$6\sin x - x^2\sin x - 4x\cos x - 2x > 0, \quad x \in \left(0, \frac{\pi}{2}\right]. \tag{5}$$

Now consider the function $g : (0, \frac{\pi}{2}] \longrightarrow R$ defined by

$$g(x) = \frac{(\sin x)\pi^3 + 16x^3 - 4\pi x^3 + 4x^2\pi^2 - x\pi^3 - 12\pi x^2}{x^2\pi^3}.$$

We have

$$g'(x) = \frac{x(\cos x)\pi^3 + 16x^3 - 4\pi x^3 + x\pi^3 - 2(\sin x)\pi^3}{x^3\pi^3}$$

and

$$g''(x) = \frac{-x^2\sin x - 4x\cos x - 2x + 6\sin x}{x^4}.$$

From (5) it follows that $g''(x) > 0$, for all $x \in (0, \frac{\pi}{2}]$. Hence $g'(x)$ is strictly increasing on $(0, \frac{\pi}{2}]$. Since $g'(\pi/2) = 0$, it follows that $g'(x) < 0$ for all $x \in (0, \frac{\pi}{2})$. Therefore $g(x)$ is strictly decreasing on $(0, \frac{\pi}{2}]$. We have $g(\pi/2) = 0$. Hence $g(x) \geq 0$ for all $x \in (0, \frac{\pi}{2}]$. Multiplying $g(x)$ by x we get

$$\begin{aligned} xg(x) &= \frac{1}{x}\frac{(\sin x)\pi^3 + 16x^3 - 4\pi x^3 + 4x^2\pi^2 - x\pi^3 - 12\pi x^2}{\pi^3} \\ &= \frac{\sin x}{x} - \frac{2}{\pi} - \frac{1}{\pi^3}(\pi^2 - 4x^2) - \frac{4(\pi - 3)}{\pi^3}\left(x - \frac{\pi}{2}\right)^2. \end{aligned}$$

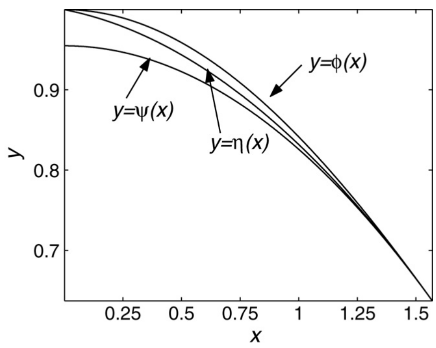

Fig. 1.

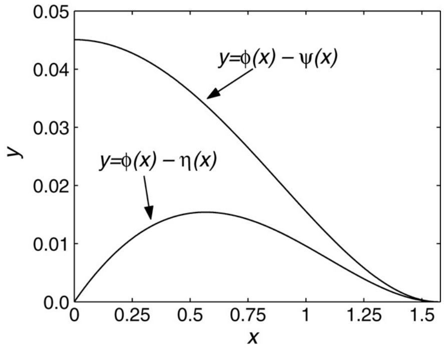

Fig. 2.

Since $x > 0$ and $g(x) \geq 0$ for all $x \in (0, \frac{\pi}{2}]$, it follows that $\frac{\sin x}{x} - \frac{2}{\pi} - \frac{1}{\pi^3}(\pi^2 - 4x^2) - \frac{4(\pi-3)}{\pi^3}(x - \frac{\pi}{2})^2 \geq 0$. So the proof is complete. $\square$

The graphs of the functions $\phi(x) = \frac{\sin x}{x}$, $\psi(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)$ and $\eta(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi-3)}{\pi^3}(x - \frac{\pi}{2})^2$ are given in Fig. 1 while the graphs of the error functions $\phi - \psi$ and $\phi - \eta$ are given in Fig. 2.

2. Applications

Yang's inequality [5] and its generalizations which play an important role in the theory of distribution of values of functions [5–7] can be stated as follows.

If $A > 0$, $B > 0$, $A + B \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$\cos^2 \lambda A + \cos^2 \lambda B - 2 \cos \lambda A \cos \lambda B \cos \lambda\pi \geq \sin^2 \lambda\pi,$$

where the equality holds if and only if $\lambda = 0$ or $A + B = \pi$. Using a generalization of Yang's inequality and the refinement (3) of Jordan's inequality, the following theorem has been developed [8].

Theorem 2.1. *Let $n \geq 2$ be a natural number, $A_i > 0$ $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} A_i \leq \pi$, and $0 \leq \lambda \leq 1$. Then*

$$R(\lambda) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq T(\lambda) \tag{6}$$

where

$$H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2\cos \lambda A_i \cos \lambda A_j \cos \lambda\pi \tag{7}$$

and

$$R(\lambda) = \frac{n(n-1)}{2}\lambda^2(3-\lambda^2)^2 \cos^2 \frac{\lambda}{2}\pi, \quad T(\lambda) = \frac{n(n-1)}{2}\lambda^2\pi^2. \tag{8}$$

In a similar way, based on the inequality (4) we can give the following.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, we have*

$$S(\lambda) \le \sum_{1\le i<j\le n} H_{ij} \le T(\lambda) \tag{9}$$

where

$$S(\lambda) = \frac{n(n-1)}{2}\lambda^2(\pi + (6-2\pi)\lambda + (\pi-4)\lambda^2)^2 \cos^2 \frac{\lambda}{2}\pi. \tag{10}$$

Proof. Substituting $x = \frac{\lambda}{2}\pi$ in (4) we get

$$\sin \frac{\lambda}{2}\pi \ge \frac{\lambda}{2}(\pi + (6-2\pi)\lambda + (\pi-4)\lambda^2).$$

Let $k(\lambda) = \pi + (6-2\pi)\lambda + (\pi-4)\lambda^2$. Then $k'(\lambda) = (6-2\pi) + (2\pi-8)\lambda < 0$, $0 \le \lambda \le 1$ and since $k(\frac{\pi}{2}) = 4\pi - 2\pi^2 + \frac{1}{4}\pi^3 > 0$ it follows that $k(\lambda) > 0$, $0 \le \lambda \le 1$ since k is decreasing on $(0, 1)$. So

$$0 \le \frac{\lambda}{2}(\pi + (6-2\pi)\,\lambda + (\pi-4)\,\lambda^2) \le \sin \frac{\lambda}{2}\pi \le \frac{\lambda}{2}\pi$$

and hence

$$\frac{\lambda^2}{4}(\pi + (6-2\pi)\lambda + (\pi-4)\lambda^2)^2 \le \sin^2 \frac{\lambda}{2}\pi \le \frac{\lambda^2}{4}\pi^2$$

or

$$\frac{\lambda^2}{4}(\pi + (6-2\pi)\,\lambda + (\pi-4)\,\lambda^2)^2 \cos^2 \frac{\lambda}{2}\pi \le \sin^2 \frac{\lambda}{2}\pi \cos^2 \frac{\lambda}{2}\pi \le \frac{\lambda^2}{4}\pi^2.$$

Since

$$\sin^2 \lambda\pi = 4\sin^2 \frac{\lambda}{2}\pi \cos^2 \frac{\lambda}{2}\pi, \tag{11}$$

using the inequality (see [7])

$$\sin^2 \lambda\pi \le H_{ij} \le 4\sin^2 \frac{\lambda}{2}\pi, \tag{12}$$

and the identity (11) it follows that

$$\lambda^2(\pi + (6-2\pi)\lambda + (\pi-4)\lambda^2)^2 \cos^2 \left(\frac{\lambda}{2}\pi\right) \le \sin^2 \lambda\pi \le H_{ij} \le \lambda^2\pi^2. \tag{13}$$

If $1 \le i < j \le n$, then (9) immediately follows from (13) since

$$\sum_{1\le i<j\le n} \lambda^2(\pi + (6-2\pi)\lambda + (\pi-4)\lambda^2)^2 \cos^2 \frac{\lambda}{2}\pi = S(\lambda)$$

Table 1
Comparison among the approximations $P(\lambda)$, $R(\lambda)$, $S(\lambda)$ and the lower bound $\sum_{1\le i<j\le n} \sin^2 \lambda\pi$

	λ	$\sum_{1\le i<j\le n} \sin^2 \lambda\pi$	$P(\lambda)$	$R(\lambda)$	$S(\lambda)$
$n = 6$	0.1	**1.432372**	0.5853169	1.308198	**1.410482**
	0.5	**15.00000**	7.500000	14.17968	**14.54708**
	0.8	**5.182372**	3.666873	5.105754	**5.130290**
	0.95	**0.3670761**	0.3333381	0.3666308	**0.3667546**
$n = 20$	0.1	**18.14338**	7.414014	16.57050	**17.86610**
	0.5	**190.0000**	95.00000	179.6093	**184.2630**
	0.8	**65.64338**	46.44706	64.67289	**64.98368**
	0.95	**4.649630**	4.222283	4.643990	**4.645558**

and

$$\sum_{1\le i<j\le n} \lambda^2\pi^2 = T(\lambda).$$

Thus the proof is complete. $\square$

Remark 2.1. It has been shown that (see [8])

$$P(\lambda) \le R(\lambda) \tag{14}$$

where

$$P(\lambda) = \sum_{1\le i<j\le n} 4\lambda^2 \cos^2 \frac{\lambda}{2}\pi = \frac{n(n-1)}{2} 4\lambda^2 \cos^2 \frac{\lambda}{2}\pi.$$

On the other hand, for $0 \le \lambda \le 1$ we have

$$0 \le (\pi - 3) - 2(\pi - 3)\lambda + (\pi - 3)\lambda^2$$

or

$$3 - \lambda^2 \le \pi + (6 - 2\pi)\lambda + \lambda^2(\pi - 4).$$

Hence

$$\lambda^2(3 - \lambda^2)^2 \le \lambda^2(\pi + (6 - 2\pi)\lambda + \lambda^2(\pi - 4))^2. \tag{15}$$

So from (12), (14) and (15) it easily follows that

$$P(\lambda) \le R(\lambda) \le S(\lambda) \le \sum_{1\le i<j\le n} \sin^2 \lambda\pi \le \sum_{1\le i<j\le n} H_{ij}, 0 \le \lambda \le 1$$

which shows that the inequality (9) we have proved is a strengthened version of the inequalities given by Debnath and Zhao in [8].

To finish, we supply Table 1, which enables us to compare the values of $\sum_{1\le i<j\le n} \sin^2 \lambda\pi$ and the values of lower bounds $P(\lambda)$, $R(\lambda)$ and $S(\lambda)$ for some values of n and λ.

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